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VISCOELASTIC RESPONSE OF A SIMPLE STRAND

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Abstract—A method is presented in which the axial viscoelastic response of a simple strand may be predicted given the stress relaxation of the filament's construction material. This approach utilizes the Schapery collocation method to determine the coefficients for the elements of a Wiechert model. The geometric effects of the strand are then combined with the analytical solution for the Wiechert model to develop a system of convolution integrals which satisfy the equilibrium and boundary conditions for the strand construction. The solutions for these integrals are approximated numerically using a modified Newton's iterative method combined with a numerical technique developed by Lee and Rogers.

INTRODUCTION

Multiple filament cords composed of twisted polymer filaments are utilized in many of today's manufactured goods. They are the main components in most textile products and the reinforcing components in many composite structures (i.e. tires, biomedical devices, etc.). The advantage of the multiple filament cord over a solid filament with an equivalent radius is the lower bending stiffness while maintaining a high axial stiffness.

Since the simple strand (see Fig. 1) is a fundamental building component of many cords with complex cross-sections, an understanding of its mechanical response is necessary

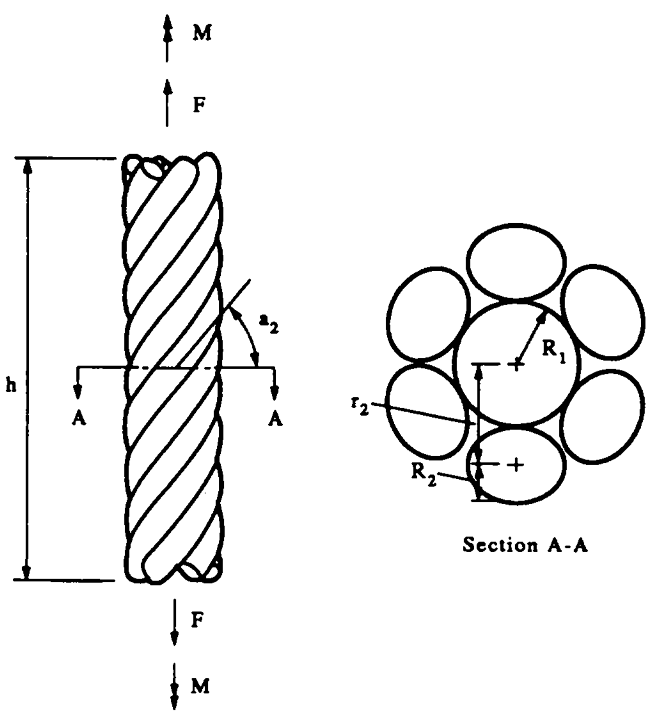

Fig. 1. Side view and cross-section of an undeformed simple strand.

if one is to understand the response of cords with more complex cross-sections. The elastic mechanical response of a simple strand has been analysed by Costello (1990) where he has developed a set of constitutive equations which describe the strand's elastic response to a variety of static and dynamic loading conditions. Conway and Costello (1991) further analysed the axial mechanical response of a simple strand. They demonstrated, analytically, that a significant reduction in contact stresses between strand wires could be achieved by a slight modification of the wire's cross-sectional geometry.

As is evident from the literature, an extensive development of the elastic simple strand model presently exists. The viscoelastic model of twisted filament cords which incorporate the time-dependent material behavior for polymer filaments has also been developed (Huang, 1978a, b). This paper presents a theory capable of predicting, within limits, the mechanical response of a simple strand composed of a polymeric material. This is accomplished by using a method similar to that described by Conway and Costello (1992). First, a viscoelastic model that incorporates relatively realistic material properties is generated. The model is, then, integrated into the constitutive equations for a simple strand (Costello, 1990). Finally, the mechanical response of the strand is evaluated numerically.

THEORY

Most polymer strand applications require that the strand should not retain any residual strains after a period of time subsequent to the removal of the applied load. For this reason, the analytical model chosen to describe the strand material's response to loading must represent a viscoelastic solid. If it is assumed that the material behaves in a linearly viscoelastic manner, a number of analytical models are available to describe the material's mechanical response. This assumption is valid for relatively small strains and will be used throughout this work (Bland, 1960). Further, it is assumed that the material is slightly compressible and functions in an isothermal environment, thus any changes in the mechanical properties due to a temperature change are not considered.

The authors of this paper have chosen to use a modified generalized Maxwell model, known as a Wiechert model (see Fig. 2), because of its relative convenience in modeling the stress relaxation for a given material resulting from an imposed strain. Since linear viscoelasticity is assumed for the material property, the response to loading for each element in the Wiechert model is linear and is described by a constant coefficient. The values for

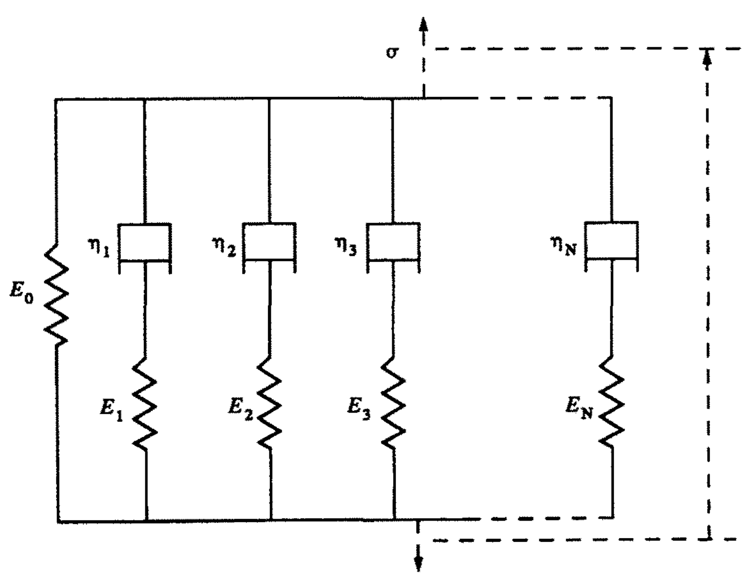

Fig. 2. Wiechert model.

these coefficients are calculated from the experimental stress relaxation data by using the Schapery Collocation method.

The Schapery Collocation method models each of the material's major transitions as a single relaxation time process (see Fig. 3). Thus the values for the stress–strain relationship

$$\sigma(t) = \left\{E_0 + \sum_{j=1}^{N} E_j \exp\left(\frac{-t}{\tau_j}\right)\right\}\varepsilon_0 \tag{1}$$

can be determined where E_0 is the rubbery modulus, E_j is the modulus for each successive transition, τ_j is the relaxation time for the corresponding transition and ε_0 is the imposed constant strain.

Once the strand's material has been mechanically characterized, the geometric constraints for the strand must be determined. The longitudinal and lateral cross-section views of an undeformed simple strand are illustrated in Fig. 1. The core filament has a radius, R_1, and each of the six outer filaments has a radius, R_2. The subscripts refer to the layer of filaments, with the first layer being the core filament.

From the geometric description of an outer filament, it is evident that the minor ellipse axis length is R_2. Thus, the helix radius for the undeformed strand is

$$r_2 = R_1 + R_2 \tag{2}$$

and the helix radius for the deformed strand is

$$r_2(t) = R_1(t) + R_2(t). \tag{3}$$

The initial helix angle, α_2, is determined from the initial pitch, P_2, of an outer filament, or

$$\alpha_2 = \tan^{-1}\frac{P_2}{2\pi r_2}. \tag{4}$$

In a purely elastic material, the time necessary for molecular rearrangement is virtually infinite (Tschoegl, 1989). A comparison of an elastic strand's initial configuration and final configuration can, therefore, be used to determine the strand's mechanical response to loading or deformation. If a load is applied to a polymer strand, however, the helix angle, $\alpha_2(t)$, each filament radius, $R_1(t)$ and $R_2(t)$, and helix radius, $r_2(t)$, all vary as a function of

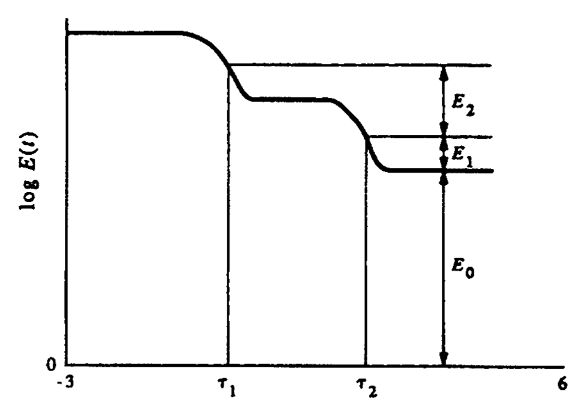

Fig. 3. Theoretical stress relaxation response.

time. Thus a description of a viscoelastic solid's final configuration is not applicable. Instead, a description of the strand's configuration at some time, $t > 0$, is necessary.

Conway and Costello (1992) have shown that for a given length of strand, it will have Q_0 turns in the undeformed state and Q turns after deformation at some time $t > 0$. The undeformed length of the strand is

$$L_0 = 2\pi Q_0 r_2 \tan \alpha_2 \tag{5}$$

and the deformed length at $t > 0$ is

$$L(t) = 2\pi Q r_2(t) \tan \alpha_2(t). \tag{6}$$

The corresponding undeformed length of a filament is

$$l_0 = 2\pi Q_0 r_2 \sec \alpha_2 \tag{7}$$

and the corresponding deformed length is

$$l(t) = 2\pi Q r_2(t) \sec \alpha_2(t). \tag{8}$$

The axial strain in the strand is thus,

$$\varepsilon_1(t) = \frac{L(t)}{L_0} - 1 = \frac{Q r_2(t) \tan \alpha_2(t)}{Q_0 r_2 \tan \alpha_2} - 1 \tag{9}$$

and the axial strain in a filament is

$$\xi_2(t) = \frac{l(t)}{l_0} - 1 = \frac{Q r_2(t) \cos \alpha_2}{Q_0 r_2 \cos \alpha_2(t)} - 1. \tag{10}$$

From Poisson's effect,

$$\frac{R_2(t)}{R_2} - 1 = -\nu \xi_2(t), \tag{11}$$

where ν is Poisson's ratio and is considered to be constant for relatively small deformations.

A combination of eqns (9), (10) and (11) results in

$$\varepsilon_1(t) = \frac{\sin \alpha_2(t)}{\sin \alpha_2}\left[1 + \frac{1}{\nu}\left(1 - \frac{R_2(t)}{R_2}\right)\right] - 1. \tag{12}$$

The angle of twist per unit length for the strand is

$$\tau_c(t) = \frac{2\pi(Q - Q_0)}{L_0} = \frac{1}{r_2 \tan \alpha_2}\left(\frac{Q}{Q_0} - 1\right). \tag{13}$$

By combining eqns (10), (11) and (13), the twist per unit length becomes

$$\tau_c(t) = \frac{1}{r_2 \tan \alpha_2}\left\{\frac{r_2 \cos \alpha_2(t)}{r_2(t) \cos \alpha_2}\left[1 + \frac{1}{\nu}\left(1 - \frac{R_2(t)}{R_2}\right)\right] - 1\right\}. \tag{14}$$

Equation (14) shows that if the strand ends are allowed to rotate, the strain will be coupled with a twist. If, however, the ends of the strand are not allowed to rotate, such that $\tau_c(t) = 0$, then an additional constrained condition exists. From eqn (14), this constraint is

$$\frac{\cos\alpha_2}{r_2} = \frac{\cos\alpha_2(t)}{r_2(t)}\left[1+\frac{1}{v}\left(1-\frac{R_2(t)}{R_2}\right)\right]. \tag{15}$$

By incorporating eqns (2) and (15), a relationship between the helix angle, $\alpha_2(t)$, and the outer filament radius, $R_2(t)$, can be shown to be

$$\cos\alpha_2(t) = \frac{(R_1(t)+R_2(t))\cos\alpha_2}{(R_1+R_2)\left[1+\frac{1}{v}\left(1-\frac{R_2(t)}{R_2}\right)\right]}. \tag{16}$$

The general design of a simple strand minimizes the contact stresses between the outer filaments by requiring the outer radius, R_2, to be less than the core radius, R_1. The reduction allows the outer filament to only have contact with the core, creating a line load between the outer filament and the core. This reduction in radius does not have to be very large, usually on the order of 3% for a polymer with a Poisson's ratio near $\frac{1}{2}$. This now allows a relationship between $R_1(t)$ and $R_2(t)$ to be determined.

Once the geometric constraints for the simple strand are determined, the equilibrium equations can be developed. Before the viscoelastic material properties are incorporated into this configuration, however, a review of the elastic equilibrium equations for a simple strand are necessary.

Costello (1990) has shown that for an axially loaded, simple strand, the outer filaments deform from one helical configuration to another. Thus, only two equations must be satisfied for each wire to be in equilibrium. These are:

$$N'_2\tau_2 - T_2\kappa'_2 - X_2 = 0 \tag{17}$$

and

$$G'_2\tau_2 - H_2\kappa'_2 + N_2 = 0, \tag{18}$$

where the subscript 2, again, refers to the outer layer of filaments, N'_2 is the binormal component of the shear load, τ_2 is the twist per unit length of the wire's centerline, T_2 is the wire's axial load, κ'_2 is the binormal component of curvature, X_2 is the resultant line load per unit length of the wire's centerline, G'_2 is the binormal component of the bending moment and H_2 is the twisting moment. Further, Costello (1990) showed that

$$G'_2 = \frac{\pi R_2^4}{4} E\left[\frac{\cos^2\bar{\alpha}_2}{\bar{r}_2} - \frac{\cos^2\alpha_2}{r_2}\right] \tag{19}$$

and

$$H_2 = \frac{\pi R_2^4}{4(1+v)} E\left[\frac{\sin\bar{\alpha}_2\cos\bar{\alpha}_2}{\bar{r}_2} - \frac{\sin\alpha_2\cos\alpha_2}{r_2}\right], \tag{20}$$

where R_2 is the outer wire radius after loading, E is the wire material's elastic modulus, $\bar{\alpha}_2$ is the helix angle of the loaded wire and $\bar{r}_2$ is the helix radius of the loaded strand. Also in the elastic case, the axial load may be determined from

$$T_2 = \frac{\pi R_2^2}{v} E\left[1 - \frac{R_2}{(R_2)_0}\right]. \tag{21}$$

Equations (2), (19) and (20) can be used to determine an expression for N'_2 in terms of $\bar{\alpha}_2$ and R_2 given v, α_2, $(R_2)_0$ and E. Subsequently, this equation for N'_2 combined with eqns (2) and (21) results in an expression for X_2 in terms of $\bar{\alpha}_2$ and R_2.

The total force, F_2, and moment, M_2, for the outer layer of wires are

$$F_2 = 6[T_2 \sin\alpha_2 + N'_2 \cos\alpha_2] \tag{22}$$

and

$$M_2 = 6[H_2 \sin\alpha_2 + G'_2 \cos\alpha_2 + T_2 r_2 \cos\alpha_2 - N'_2 r_2 \sin\alpha_2]. \tag{23}$$

The total force, F_1, in the core wire is calculated similarly to the axial load in an outer wire where

$$F_1 = -\frac{\pi R_1^2}{\nu} E\left[1 - \frac{R_1}{(R_1)_0}\right]. \tag{24}$$

The total moment, M_1, in the core wire is equal to zero as a result of the no end rotation boundary condition imposed on the simple strand.

Finally, the total axial load, F_T, and the total moment, M_T, in the simple strand are

$$F_T = F_1 + F_2 \tag{25}$$

and

$$M_T = 0 + M_2. \tag{26}$$

Returning now to the time dependent behavior of the polymer strand, according to the correspondence principle of viscoelasticity, the tensile relaxation modulus can be treated as an integral operation. The correspondence principle allows Hooke's law to be used in the Laplace transform space and is represented as

$$\bar{\sigma}(p) = p\bar{E}(p)\bar{\varepsilon}(p), \tag{27}$$

where p is the transform variable. Inversion of this transform yields an expression for the Boltzmann superposition integral (Tschoegl, 1990). This general solution to the inverse transform of eqn (27) is

$$\sigma(t) = \int_0^t E(t-t_1)\frac{\mathrm{d}\varepsilon(t_1)}{\mathrm{d}t_1}\,\mathrm{d}t_1. \tag{28}$$

Now, instead of varying the modulus and the strain, we now hold the strain constant, as in stress relaxation, and allow another parameter of the design to vary. The convolution integral would have the general form

$$\int_0^t E(t-t_1)\dot{f}(t_1)\,\mathrm{d}t_1. \tag{29}$$

If the differential term is denoted as

$$df = \dot{f}, \tag{30}$$

then eqn (29) can be represented as

$$(E * \mathrm{d}f)(t). \tag{31}$$

The time-dependent relationships for the binormal component of the bending moment, $G'_2(t)$, and the twisting moment, $H_2(t)$, for an outer filament can now be determined. These components of the moment in an outer filament become

$$G'_2(t) = \frac{\pi R_2^4(t)}{4}\left(E * \mathrm{d}\left[\frac{\cos^2\alpha_2}{r_2}\right]\right)(t) \tag{32}$$

and

$$H_2(t) = \frac{\pi R_2^4(t)}{4(1+v)}\left(E * \mathrm{d}\left[\frac{\sin\alpha_2\cos\alpha_2}{r_2}\right]\right)(t). \tag{33}$$

By incorporating eqns (18), (32) and (33), the binormal component of shear, $N'_2(t)$, is

$$N'_2(t) = \frac{\pi R_2^4(t)\cos\alpha_2(t)}{4r_2(t)}\left\{\left[\frac{\cos\alpha_2(t)}{1+v}\right]\left(E * \mathrm{d}\left[\frac{\sin\alpha_2\cos\alpha_2}{r_2}\right]\right)(t) - \sin\alpha_2(t)\left(E * \mathrm{d}\left[\frac{\cos^2\alpha_2}{r_2}\right]\right)(t)\right\}. \tag{34}$$

The axial load in an outer filament can be determined from eqn (21) to be

$$T_2(t) = -\frac{\pi R_2^2(t)}{vR_2}(E * \mathrm{d}R_2)(t). \tag{35}$$

By combining eqns (17), (34) and (35), the contact load per unit length for an outer filament is

$$X_2(t) = \frac{\pi R_2^4(t)\sin\alpha_2(t)\cos^2\alpha_2(t)}{4r_2(t)}\left\{\left[\frac{\cos\alpha_2(t)}{1+v}\right]\left(E * \mathrm{d}\left[\frac{\sin\alpha_2\cos\alpha_2}{r_2}\right]\right)(t) - \sin\alpha_2(t)\left(E * \mathrm{d}\left[\frac{\cos^2\alpha_2}{r_2}\right]\right)(t)\right\} + \frac{\pi R_2^2(t)\cos^2\alpha_2(t)}{vR_2}(E * \mathrm{d}R_2)(t). \tag{36}$$

The total force, $F_2(t)$, and moment, $M_2(t)$, for the outer layer of filaments are

$$F_2(t) = 6[T_2(t)\sin\alpha_2(t) + N'_2(t)\cos\alpha_2(t)] \tag{37}$$

and

$$M_2(t) = 6[H_2(t)\sin\alpha_2(t) + G'_2(t)\cos\alpha_2(t) + T_2(t)r_2(t)\cos\alpha_2(t) - N'_2(t)r_2(t)\sin\alpha_2(t)]. \tag{38}$$

The total force, $F_1(t)$, in the core filament is calculated similarly to the axial load in an outer filament where

$$F_1(t) = -\frac{\pi R_1^2(t)}{vR_1}(E * \mathrm{d}R_1)(t). \tag{39}$$

The total moment, $M_1(t)$, in the core filament is, again, equal to zero as a result of the no end rotation boundary condition imposed on the simple strand.

Finally, the total axial load, $F(t)$, and the total moment, $M(t)$, in the simple strand are

$$F(t) = F_1(t) + F_2(t) \tag{40}$$

and

$$M(t) = 0 + M_2(t). \tag{41}$$

With eqns (28)–(37), the mathematical tools are now available for the development of a solution technique for the response of a simple strand given a specific loading criterion.

GENERAL METHOD OF SOLUTION

Now, the parameters used to calculate the various aspects of the simple strand response must be non-dimensionalized. In the subsequent analysis, R_2 and $R_2(t)$ will be evaluated in terms of R_1 and $R_1(t)$. Also, the tensile modulus at time, $t = 0^+$, is $E(0^+)$ and is denoted by E_0. Thus, all of the variables can be non-dimensionalized in terms of E_0 and R_1. This results in the following variables where the bar indicates a dimensionless quantity:

$$\begin{gathered}
\bar{E} = \frac{E(t)}{E_0}, \quad \bar{R}_1(t) = \frac{R_1(t)}{R_1}, \quad \bar{R}_2(t) = \frac{R_2(t)}{R_1}, \\
\bar{R}_1 = 1, \quad \bar{R}_2 = \frac{R_2}{R_1}, \quad \bar{r}_2(t) = \frac{r_2(t)}{R_1}, \\
\bar{T}_2(t) = \frac{T_2(t)}{E_0\pi R_1^2}, \quad \bar{N}'_2(t) = \frac{N'_2(t)}{E_0\pi R_1^2}, \quad \bar{X}_2(t) = \frac{X_2(t)}{E_0\pi R_1}, \\
\bar{G}'_2(t) = \frac{G'_2(t)}{E_0\pi R_1^3}, \quad \bar{H}_2(t) = \frac{H_2(t)}{E_0\pi R_1^3}, \quad \bar{F}_1(t) = \frac{F_1(t)}{E_0\pi R_1^2}, \\
\bar{F}_2(t) = \frac{F_2(t)}{E_0\pi R_1^2}, \quad \bar{M}_2(t) = \frac{M_2(t)}{E_0\pi R_1^3}, \quad \bar{F}(t) = \frac{F(t)}{E_0\pi R_1^2}, \quad \bar{M}(t) = \frac{M(t)}{E_0\pi R_1^3}.
\end{gathered} \tag{42}$$

For specific values of v, $F(t)$ and $E(t)$, the viscoelastic deformation of a simple strand is governed by the following equations:

$$\bar{F}_1(t) = -\frac{1}{v}\bar{R}_1^2(t)(\bar{E} * \mathrm{d}\bar{R}_1)(t), \tag{43}$$

$$\bar{T}_2(t) = -\frac{1}{v}\bar{R}_2^2(t)(\bar{E} * \mathrm{d}\bar{R}_2)(t), \tag{44}$$

$$\begin{aligned}
\bar{N}'_2(t) = \frac{\bar{R}_2^4(t)\cos\alpha_2(t)}{4\bar{r}_2(t)} \Bigg\{ & \left[\frac{\cos\alpha_2(t)}{1+v}\right]\left(\bar{E} * \mathrm{d}\left[\frac{\sin\alpha_2\cos\alpha_2}{\bar{r}_2}\right]\right)(t) \\
& - \sin\alpha_2(t)\left(\bar{E} * \mathrm{d}\left[\frac{\cos^2\alpha_2}{\bar{r}_2}\right]\right)(t)\Bigg\},
\end{aligned} \tag{45}$$

$$\begin{aligned}
\bar{X}_2(t) = \frac{\bar{R}_2^4(t)\sin\alpha_2(t)\cos^2\alpha_2(t)}{4\bar{r}_2(t)} \Bigg\{ & \left[\frac{\cos\alpha_2(t)}{1+v}\right]\left(\bar{E} * \mathrm{d}\left[\frac{\sin\alpha_2\cos\alpha_2}{\bar{r}_2}\right]\right)(t) \\
& - \sin\alpha_2(t)\left(\bar{E} * \mathrm{d}\left[\frac{\cos^2\alpha_2}{\bar{r}_2}\right]\right)(t)\Bigg\} + \frac{\bar{R}_2^2(t)\cos^2\alpha_2(t)}{v\bar{R}_2}(\bar{E} * \mathrm{d}\bar{R}_2)(t),
\end{aligned} \tag{46}$$

$$\bar{G}'_2(t) = \frac{\bar{R}_2^4(t)}{4}\left(\bar{E} * \mathrm{d}\left[\frac{\cos^2\alpha_2}{\bar{r}_2}\right]\right)(t), \tag{47}$$

$$\bar{H}_2(t) = \frac{\bar{R}_2^4(t)}{4(1+v)}\left(\bar{E} * \mathrm{d}\left[\frac{\sin\alpha_2\cos\alpha_2}{\bar{r}_2}\right]\right)(t), \tag{48}$$

$$\phi = \bar{F}(t) - \{\bar{F}_1(t) + 6[\bar{T}_2(t)\sin\alpha_2(t) + \bar{N}'_2(t)\cos\alpha_2(t)]\}, \tag{49}$$

$$\bar{M}(t) = 6[\bar{H}_2(t)\sin\alpha_2(t) + \bar{G}'_2(t)\cos\alpha_2(t) + \bar{T}_2(t)\bar{r}_2(t)\cos\alpha_2(t) - \bar{N}'_2(t)\bar{r}_2(t)\sin\alpha_2(t)]. \tag{50}$$

When the external force, $F(t)$, and a boundary condition such as no end rotation are applied at $t = 0$, the instantaneous response at $t = 0^+$ is elastic. The governing equations for this elastic response are:

$$\bar{F}_1(0^+) = \frac{1}{\nu}\bar{R}_1^2(0^+)[1 - R_1(0^+)], \tag{51}$$

$$\bar{T}_2(0^+) = \frac{1}{\nu}\bar{R}_2^2(0^+)[\bar{R}_2 - R_2(0^+)], \tag{52}$$

$$\bar{N}'_2(0^+) = \frac{\bar{R}_2^4(0^+)\cos\alpha_2(0^+)}{4\bar{r}_2(0^+)}\left\{\left[\frac{\cos\alpha_2(0^+)}{1+\nu}\right]\left[\frac{\sin\alpha_2(0^+)\cos\alpha_2(0^+)}{\bar{r}_2(0^+)} - \frac{\sin\alpha_2\cos\alpha_2}{\bar{r}_2}\right] - \sin\alpha_2(0^+)\left[\frac{\cos^2\alpha_2(0^+)}{\bar{r}_2(0^+)} - \frac{\cos^2\alpha_2}{\bar{r}_2}\right]\right\}, \tag{53}$$

$$\phi = \bar{F}(0^+) - \{\bar{F}_1(0^+) + 6[\bar{T}_2(0^+)\sin\alpha_2(0^+) + \bar{N}'_2(0^+)\cos\alpha_2(0^+)]\}. \tag{54}$$

In order to find the correct value for $\bar{R}(0^+)$ in eqns (51)–(53), an iterative method must be used to satisfy eqn (54). As stated previously, the analysis of a simple strand is limited to a no end rotation boundary condition. Thus, a modified Newton's iterative method is used. This method uses three values of $\bar{R}(0^+)$. These are $\bar{R}_i(0^+)$, $\bar{R}_i(0^+) - \Delta$ and $\bar{R}_i(0^+) + \Delta$, where Δ is a small number. From these values the corresponding terms ϕ_1, ϕ_2 and ϕ_3 can be calculated. The derivative $\mathrm{d}\phi/\mathrm{d}\bar{R}$ at $\bar{R}(0^+) = \bar{R}_i(0^+)$ can be approximated by using a central difference equation, such as

$$\phi' = \left.\frac{\mathrm{d}\phi}{\mathrm{d}\bar{R}}\right|_{\bar{R}=\bar{R}_i} = \frac{1}{2\Delta}(\phi_3 - \phi_2). \tag{55}$$

Now, Newton's iterative formula can be used to determine a new value for $\bar{R}(0^+)$ which is

$$\bar{R}_{i+1} = \bar{R}_i - \frac{\phi_1(0^+)}{\phi'} = \frac{2\Delta\phi_1(0^+)}{\phi_3(0^+) - \phi_2(0^+)}. \tag{56}$$

This iteration technique is used until $\phi_1(0^+) \approx 0$. Since this technique is used on a computer, a minimum value for $\phi_1(0^+)$, such as 1×10^{-9}, must be set. Once this value is arrived at, the corresponding value for $\bar{R}(0^+)$ is used to determine the loads and moments at $t = 0^+$.

Once the filaments in a strand are deformed into a new helical configuration by this elastic response, they continue to vary with respect to loading and deformation over time, depending on the viscoelastic properties of the strand construction material. This time-dependent behavior can be closely approximated by using a numerical integration technique developed by Lee and Rogers (1963) which evaluates the hereditary integral introduced in eqn (29). This technique uses the equation

$$\int_0^t \bar{E}(t-t_1)\dot{f}(t_1)\,\mathrm{d}t_1 = S_N + \tfrac{1}{2}[1+\bar{E}(t_N-t_{N-1})]f(t_N), \tag{57}$$

where

$$S_N = \bar{E}(t_N)[f(0^+)-f_0] - \tfrac{1}{2}[1+\bar{E}(t_N-t_{N-1})]f(t_{N-1}) \times \frac{1}{2}\sum_{i=0}^{N-2}\{[\bar{E}(t_N-t_{i+1})+\bar{E}(t_N-t_i)][f(t_{i+1})-f(t_i)]\}. \tag{58}$$

In this numerical scheme the time, t, is divided into N intervals where t_0 is the time of the initial elastic response and t_N is the time when the viscoelastic response is to be evaluated. Also, f is the function that is differentiated with respect to time in eqns (32)–(36). At each step in the time interval, a value for the convolution integral is determined from this method. This value is used to determine values for $\bar{N}'_2(t)$ and $\bar{T}_2(t)$ from eqns (34) and (35). $\bar{N}'_2(t)$ and $\bar{T}_2(t)$ are then placed in eqn (49). If ϕ is not less than some prescribed value, the modified Newton's iterative method is used to evaluate $\bar{R}_i(t)$ which will satisfy the conditions for ϕ. Once a value for $\bar{R}_i(t)$ is determined, $\alpha_2(t)$ can be calculated from eqn (16). This can then be used to determine $F_1(t)$, $\bar{T}_2(t)$, $N'_2(t)$, $\bar{X}_2(t)$, $\bar{G}'_2(t)$, $\bar{H}_2(t)$, $\bar{M}_2(t)$ and $\varepsilon_1(t)$ from eqns (43)–(48), (50) and (12), for the specific time interval from which $\bar{R}_i(t)$ was evaluated. This iterative technique must then be used at each time interval in order to describe, satisfactorily, the viscoelastic response for the strand.

STRAND RESPONSE FOR A PARTICULAR VISCOELASTIC MATERIAL

In order to determine the time-dependent response of a polymer strand, a viscoelastic material must be chosen. This material must have a linear stress–strain relationship in the Laplace transform space since the correspondence principle in viscoelasticity is used in the subsequent analysis. A material which meets this criterion is polymethyl methacrylate (PMMA). The modulus relaxation is shown in Fig. 4. Also in this figure is the corresponding theoretical curve determined from the Schapery collocation method. As discussed previously, the Wiechert model is used in the prediction of this theoretical curve. For good correlation between the experimental data for PMMA and the theoretical results, 10 Maxwell elements are required along with a spring in parallel. The modulus relaxation equation is thus

$$E(t) = 2.24\times10^6 + 1.60\times10^9\,e^{(-t/0.01)} - 7.01\times10^8\,e^{(-t/0.1)} + 9.65\times10^8\,e^{(-t)} + 4.36\times10^8\,e^{(-t/10)} + 4.12\times10^8\,e^{(-t/100)} + 2.47\times10^8\,e^{(-t/1000)} + 4.98\times10^7\,e^{(-t/10000)} + 1.27\times10^7\,e^{(-t/100000)} + 5.85\times10^6\,e^{(-t/1000000)} + 1.94\times10^6\,e^{(-t/100000000)}, \tag{59}$$

where at $t > 10^9$ hours, $E(t) \sim 10^{6.35}$ N m^{-2}.

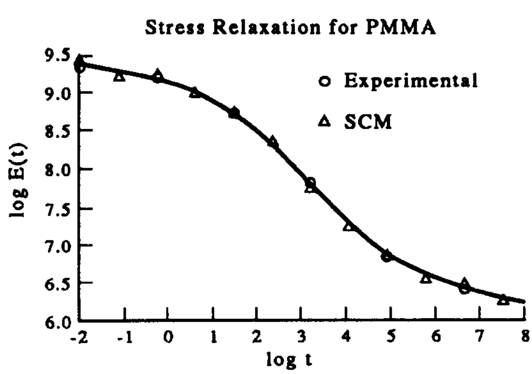

Fig. 4. Stress relaxation for polymethyl methacrylate (Lee and Rogers, 1963).

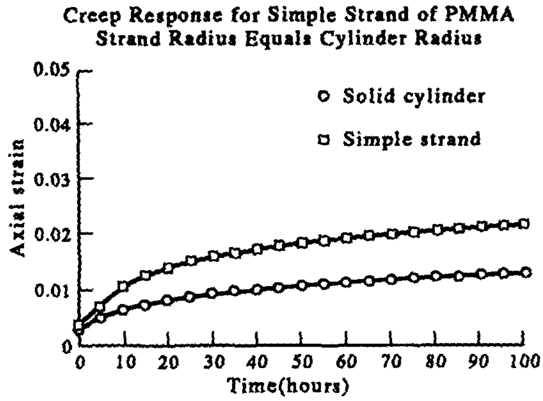

Fig. 5. Creep response for simple strand of PMMA.

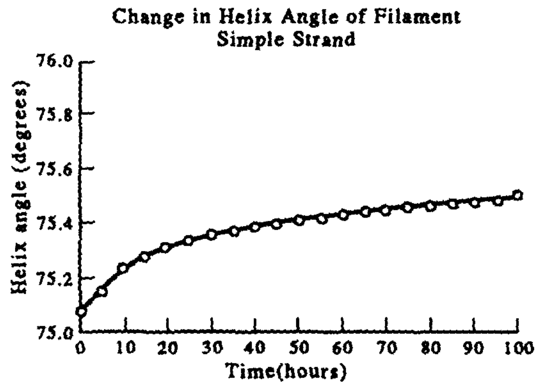

Fig. 6. Change in helix angle of outer filament.

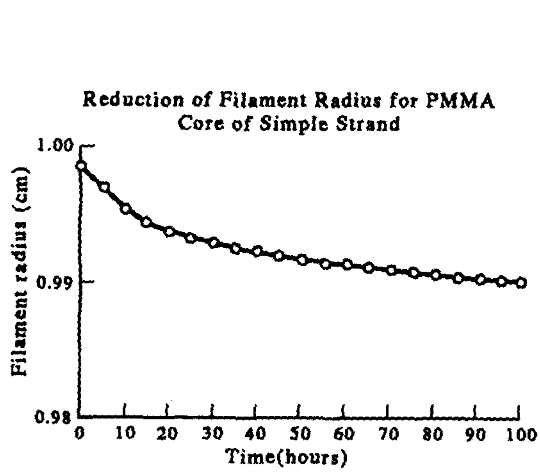

Fig. 7. Reduction of core filament radius.

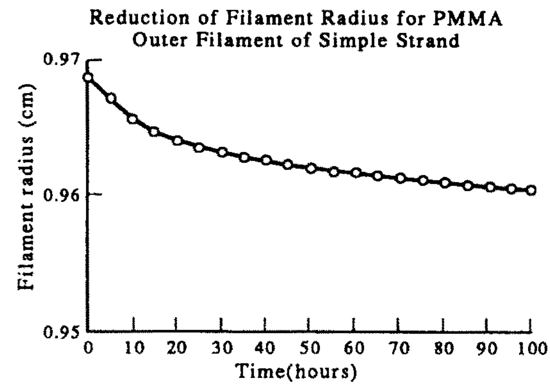

Fig. 8. Reduction of outer filament radius.

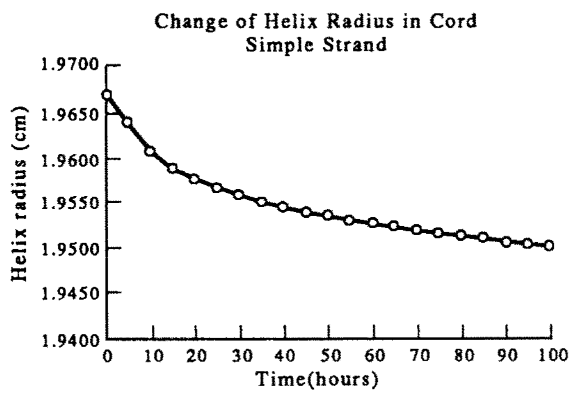

Fig. 9. Change in helix radius.

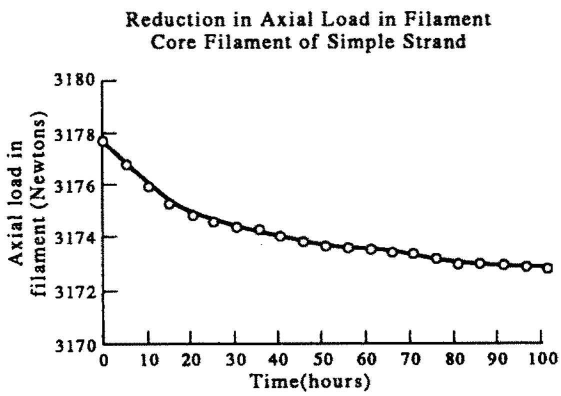

Fig. 10. Reduction of axial load in core filament.

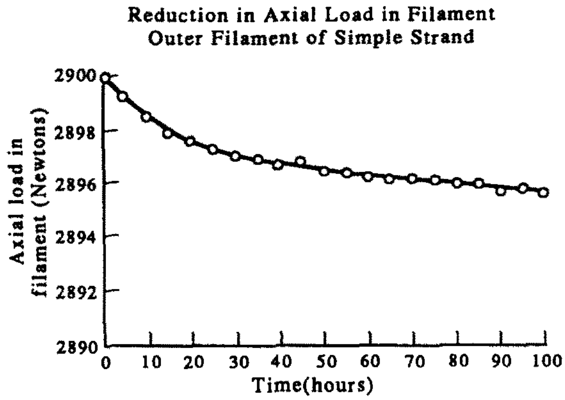

Fig. 11. Reduction of axial load in outer filament.

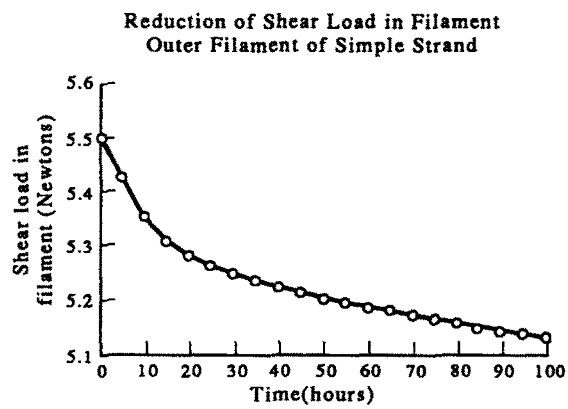

Fig. 12. Reduction of shear load in outer filament.

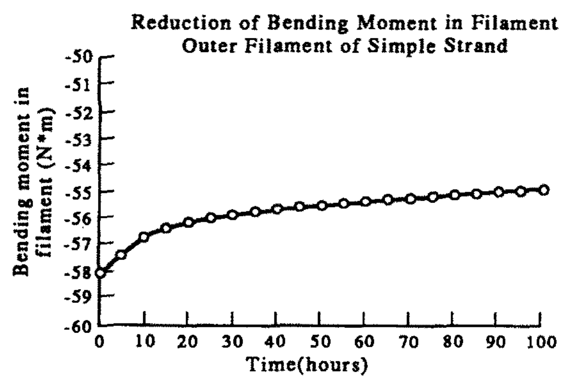

Fig. 13. Reduction of bending moment in outer filament.

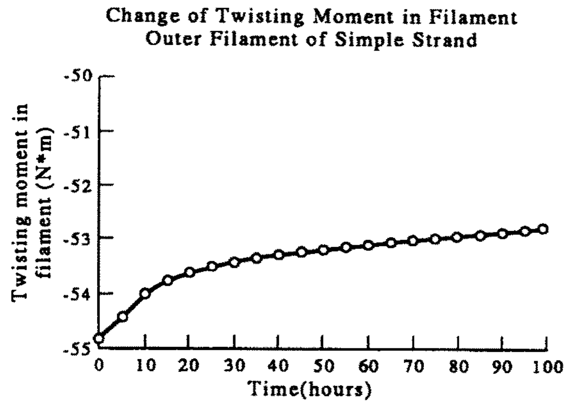

Fig. 14. Reduction of twisting moment in outer filament.

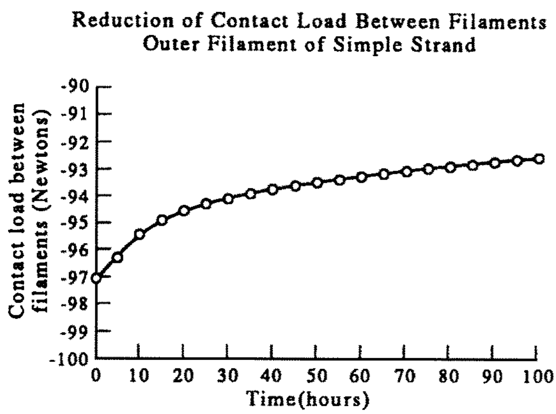

Fig. 15. Reduction of contact load between core and outer filaments.

Once the modulus relaxation is closely approximated numerically, the creep response for a solid cylindrical rod and a simple strand with equal diameters can be computed. Figure 5 shows this comparison were both the rod and the strand have an overall, initial diameter of 5.88 cm, a Poisson's ratio of 0.45 and a total axial load of 2×10^4 N. The outer layer of filaments for the simple strand have a helix angle of 75° and a radius equal to 97% of the core radius. This difference in radii ensures that during deformation the resultant contact loading is not influenced by outer filaments significantly touching each other. The no end rotation boundary condition is imposed on the strand. The instantaneous jump in strain at $t = 0^+$ is, again, caused by the initial elastic response of the material, as described in eqns (51)–(53). Both strains asymptotically approach an equilibrium value corresponding to the delayed time modulus for PMMA of $10^{6.35}$ N m^{-2}, shown in Fig. 4.

The time-dependent geometric relationships in the simple strand are shown in Figs 6–9. Figures 10–12 show the change in axial loading in the core filament and the change in axial and shear loading in an outer filament. It can be seen that the core filament carries more of the axial load than each of the outer filaments, however, the six outer filaments carry a total of 84% of the axial load in the simple strand. Note, also, that the reduction in axial load for the core and outer filament are almost identical. The large reduction in the shear load in Fig. 12 is caused by the other filaments tending to straighten out thus reducing the shear component of the load.

The bending and twisting components of the moment imposed on the outer filaments by the no end rotation boundary condition are shown in Figs 13 and 14. Note that this same boundary condition prevents a twisting moment from being imposed on the core filament. The reduction in the bending moment is much greater than for the twisting moment because, as mentioned several times, the outer filaments tend to straighten out over time, reducing the bending component of the total moment.

Since the core filament is slightly larger than the outer filaments, the resultant contact load for each outer filament is between itself and the core filament in the direction of the central axis of the helix radius. The reduction in this contact load with respect to time is shown in Fig. 15. This reduction is predominantly due to the softening of the PMMA over time and not to the straightening out of the outer filaments. The contact load between an outer filament and the core filament would, however, be reduced if the strand were allowed to rotate.

SUMMARY AND CONCLUSIONS

An analytical method is presented for determining various mechanical properties of a simple strand. These properties include the overall axial, time-dependent strain of the strand as well as the internal time-dependent, geometric and loading variations. The internal construction parameters of the strand are also included in this model.

The mechanical stress–strain relationship for a linear viscoelastic solid has been modeled by combining linear springs and dashpots into what is known as the Wiechert model. The parameters for this time-dependent model have been determined for a specific linear viscoelastic material, polymethyl methacrylate (PMMA), by using the Schapery collocation method. A numerical integration technique was then introduced to solve a convolution integral which developed, as a result of the correspondence principle of linear viscoelasticity, from the inverse Laplace transform of the elastic solution for a simple strand. This integration technique was combined with a modified Newton's iterative technique to solve the time-dependent, geometric and loading relationships in a simple strand.

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REFERENCES

Bland, D. R. (1960). *The Theory of Linear Viscoelasticity*. Pergamon Press, New York.
Conway, T. A. and Costello, G. A. (1991). Response of a strand with elliptical outer wires. *Int. J. Solids Structures* **28**(1), 33–42.
Conway, T. A. and Costello, G. A. (1992). Time-dependent response of a three filament cord. *J. Engng Mech. ASCE* (in press).
Costello, G. A. (1990). *Theory of Wire Rope*. Springer, New York.
Huang, N. C. (1978a). On the finite extension of a viscoelastic two-ply filament yarn. *Text. Res. Jl* **48**, 61–67.
Huang, N. C. (1978b). Finite extension of viscoelastic multiple yarn. *Int. J. Solids and Structures* **14**, 579–590.
Lee, E. H. and Rogers, T. G. (1963). Solution of viscoelastic stress analysis problems using measured creep or relaxation functions. *J. Appl. Mech.* **33**, 127–133.
Tschoegl, N. W. (1989). *The Phenomenological Theory of Linear Viscoelastic Behavior*. Springer, New York.